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THE MIXED FINITE ELEMENT FORMULATION FOR THREE-DIMENSIONAL BARS

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Abstract—In this study, mixed finite element equations which are based on a new functional are obtained by Gâteaux differential. This formulation is applicable to three-dimensional bars with arbitrary geometry and variable cross-sections. Boundary conditions are included in the element equations. Known nodal variable values are imposed by the Lagrange multiplier method. This newly suggested Full Functional Method (FFM) approach gives very accurate results using a few elements.

NOTATION

$I(\mathbf{y})$	functional
$\mathbf{T}, \mathbf{M}$	internal forces
$\mathbf{q}, \mathbf{m}$	external loads
$\mathbf{\Omega}, \omega$	rotation vector of bars, unit rotation vector
$\mathbf{u}$	displacement vector
$\mathbf{t}$	tangential unit vector
s	arc length along the curve
$\hat{\mathbf{T}}, \hat{\mathbf{M}}, \hat{\mathbf{\Omega}}, \hat{\mathbf{u}}$	known quantities
γ	shear vector
κ, τ	curvature and torsion of helix
R	radius of base circle
A, k'	area of cross-section, shear coefficient
I_n, I_b, I_0	moment of inertia of cross-section with respect to axes by indices
$[f, g]$	inner product
$[\ ,\ ]_\sigma$	valid at the point where dynamical condition is given
$[\ ,\ ]_\varepsilon$	valid at the point where geometric condition is given
$\mathbf{Q}$	operator
$\mathcal{L}$	coefficient matrix

1. INTRODUCTION

During the past two decades, the finite element method has become a very popular technique for computer solutions of complex problems. In the traditional finite element analysis of bars the displacements are chosen as primary variables, and the nodal value of displacements are obtained by extremizing the complementary energy functionals. This type of approach can be found in Zienkiewicz and Cheung (1970). To get more accurate results more efforts have been made. One of the approaches includes the effects of shear deformation, and is called Kirchhoff analysis as implemented by Bathe (1982). Application of this theory to the cantilever beam yields results with 75% accuracy. In the mixed finite element formulation the complementary energy function has been extended by the Lagrange multiplier method and different nodal variables are preserved independently in the functional. Recently Prathap and Babu (1986a,b) included the shear effect in the strain energy and assumed two independent variables (w, θ displacement and rotation) and found good results for straight beams. Babu and Prathap (1986) developed a method for curved bars, based on the energy method in their original work. Prathap and Babu (1986a,b) also studied thick curved beams. Two- or three-dimensional finite element formulations can also be applied to solve beam problems. Some studies for solving beam problems exist in the literature (Mirza and Olson, 1980; Spilker and Singh, 1982). However, the disadvantage of these methods is that they are time consuming. In this study the Oden–Reddy Gâteaux differential approach is used to obtain a new functional. This functional provides an elegant strategy for constructing an element matrix using a helical element to solve three-dimensional bars with arbitrary geometry and variable cross-sections. Boundary conditions are included in the element equations. The values of the known nodal variables are imposed by Lagrange Multipliers. This formulation can be easily applied to simpler problems such as plane circular beams and other plane problems.

2. FIELD EQUATIONS AND FUNCTIONAL

The field equations of bars for the three-dimensional case are given by Inan (1966) as follows:

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} - \mathbf{q} = 0 \\ -&\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s} - \mathbf{t} \times \mathbf{T} - \mathbf{m} = 0 \quad \text{Equilibrium equations} \end{aligned} \tag{1}$$

$$\begin{aligned} &\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}s} - \boldsymbol{\omega} = 0 \\ &\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} - \mathbf{t} \times \boldsymbol{\Omega} - \boldsymbol{\gamma} = 0 \quad \text{Kinematic equations} \end{aligned} \tag{2}$$

$$\begin{aligned} &-\mathbf{M} - \mathbf{D} \cdot \boldsymbol{\omega} = 0 \\ &-\mathbf{T} - \mathbf{C} \cdot \boldsymbol{\gamma} = 0 \quad \text{Constitutive equations} \end{aligned} \tag{3}$$

$$\begin{aligned} &\mathbf{T} - \hat{\mathbf{T}} = 0 \\ &\mathbf{M} - \hat{\mathbf{M}} = 0 \\ &\boldsymbol{\Omega} - \hat{\boldsymbol{\Omega}} = 0 \\ &\mathbf{u} - \hat{\mathbf{u}} = 0 \quad \text{Boundary conditions.} \end{aligned} \tag{4}$$

Bending and shear rigidities are defined as follows:

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{k'}{GA} & 0 & 0 \\ 0 & \frac{k'}{GA} & 0 \\ 0 & 0 & \frac{1}{EA} \end{bmatrix} \qquad \mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{EI_n} & 0 & 0 \\ 0 & \frac{1}{EI_b} & 0 \\ 0 & 0 & \frac{1}{EI_t} \end{bmatrix}. \tag{5}$$

The use of the Gâteaux differential method (Oden and Reddy, 1976) field equation of bars yields the following functional derived by Aköz (1985, 1987). Information about the functional is given in the Appendix.

$$\begin{aligned} I(\mathbf{y}) = &-\left[\mathbf{u}, \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right] + [\mathbf{t} \times \boldsymbol{\Omega}, \mathbf{T}] - [\mathbf{q}, \mathbf{u}] - [\mathbf{m}, \boldsymbol{\Omega}] - \left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s}, \boldsymbol{\Omega}\right] - \tfrac{1}{2}[\boldsymbol{\omega}, \mathbf{M}] - \tfrac{1}{2}[\boldsymbol{\gamma}, \mathbf{T}] \\ &- \tfrac{1}{2}[(\mathbf{M} - \mathbf{D}\boldsymbol{\omega}), \boldsymbol{\omega}] - \tfrac{1}{2}[(\mathbf{T} - \mathbf{C}\boldsymbol{\gamma}), \boldsymbol{\gamma}] + [(\mathbf{T} - \hat{\mathbf{T}}), \mathbf{u}]_\sigma + [(\mathbf{M} - \hat{\mathbf{M}}), \boldsymbol{\Omega}]_\sigma + [\hat{\mathbf{u}}, \mathbf{T}]_\varepsilon + [\hat{\boldsymbol{\Omega}}, \mathbf{M}]_\varepsilon. \end{aligned} \tag{6}$$

3. THE FINITE ELEMENT FORMULATION

In eqn (5) there are six unknown vectorial quantities: $\mathbf{T}$, $\mathbf{M}$, $\mathbf{u}$, $\mathbf{\Omega}$, ω and γ. For the sake of simplicity after ω and γ have been eliminated from the functional using eqn (3), (6) reduces to

$$I(\mathbf{y}) = -\left[\mathbf{u}, \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right] + [\mathbf{t}\times\mathbf{\Omega}, \mathbf{T}] - [\mathbf{q}, \mathbf{u}] - [\mathbf{m}, \mathbf{\Omega}] - \left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s}, \mathbf{\Omega}\right] - \tfrac{1}{2}[\mathbf{D}^{-1}\mathbf{M}, \mathbf{M}] - \tfrac{1}{2}[\mathbf{C}^{-1}\mathbf{T}, \mathbf{T}]$$
$$+ [\hat{\mathbf{u}}, \mathbf{T}]_{\varepsilon} + [\hat{\mathbf{\Omega}}, \mathbf{M}]_{\varepsilon} + [(\mathbf{T}-\hat{\mathbf{T}}), \mathbf{u}]_{\sigma} + [(\mathbf{M}-\hat{\mathbf{M}}), \mathbf{\Omega}]_{\sigma}. \quad (7)$$

In this functional $\mathbf{M}$, $\mathbf{T}$, $\mathbf{u}$ and $\mathbf{\Omega}$ are unknown vectorial quantities. The first two vectors $\mathbf{M}$ and $\mathbf{T}$ define the internal force distribution which is important in engineering design, while the last two vectors give the deformation of the structure. Here the solution domain is the general three-dimensional curve. To represent this general curve circular helix elements have been chosen (Fig. 1).

Coordinate axes are depicted in Fig. 1. $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are Frenet coordinate unit vectors. Derivations of these vectors with respect to arc lengths are given by Frenet–Serret formulas (Sokolnikoff and Redheffer, 1988):

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa\mathbf{n}, \quad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -\kappa\mathbf{t}+\tau\mathbf{b}, \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\tau\mathbf{n} \quad (8)$$

where κ and τ are curvature and torsion respectively. The position vector for a circular helix is

$$\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j} + p\theta\mathbf{k} \quad (9)$$

where p is the step for unit angle. Curvature, torsion and arc length of the helix are defined as follows:

$$\kappa = R/C^2, \quad \tau = -p/C^2, \quad \mathrm{d}s = c\,\mathrm{d}\theta \quad (10)$$

where R is the radius of the base circle and

$$C = \sqrt{R^2+p^2}. \quad (11)$$

The necessary transformations between unit vectors for cartesian and Frenet–Serret coordinate axes are:

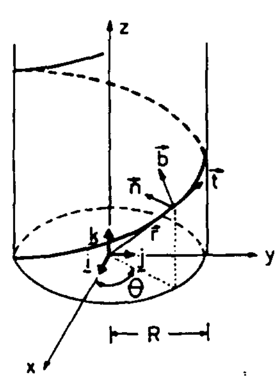


Fig. 1. Local and global coordinate axes.

$$\mathbf{i} = -(R/C)\sin\theta\mathbf{t} - \cos\theta\mathbf{n} + (p/C)\sin\theta\mathbf{b}$$
$$\mathbf{j} = (R/C)\cos\theta\mathbf{t} - \sin\theta\mathbf{n} - (p/C)\cos\theta\mathbf{b}$$
$$\mathbf{k} = (p/C)\mathbf{t} + (R/C)\mathbf{b} \tag{12}$$

and the inverse transformations are:

$$\mathbf{t} = -(R/C)\sin\theta\mathbf{i} + (R/C)\cos\theta\mathbf{j} + (p/C)\mathbf{k}$$
$$\mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$
$$\mathbf{b} = (p/C)\sin\theta\mathbf{i} - (p/C)\cos\theta\mathbf{j} + (R/C)\mathbf{k}. \tag{13}$$

The basic four unknown vectorial quantities are expressed in these coordinate axes as follows:

$$\mathbf{u} = u_t\mathbf{t} + u_n\mathbf{n} + u_b\mathbf{b}, \quad \mathbf{\Omega} = \Omega_t\mathbf{t} + \Omega_n\mathbf{n} + \Omega_b\mathbf{b}$$
$$\mathbf{T} = T_t\mathbf{t} + T_n\mathbf{n} + T_b\mathbf{b}, \quad \mathbf{M} = M_t\mathbf{t} + M_n\mathbf{n} + M_b\mathbf{b}. \tag{14}$$

For the sake of simplicity the following symbols will be adopted for the above vectorial components:

$$u_t = u, \quad u_n = v, \quad u_b = w,$$
$$\Omega_t = b, \quad \Omega_n = d, \quad \Omega_b = e,$$
$$T_t = N, \quad T_n = T, \quad T_b = Q,$$
$$M_t = B, \quad M_n = M, \quad M_b = E. \tag{15}$$

These 12 variables can be expressed by interpolation functions in the element. As an example

$$u = u_i\phi_i + u_j\phi_j \tag{16}$$

where u_i and u_j are respectively left and right nodal values of u, and ϕ_i and ϕ_j are interpolation functions. The different types of interpolation function depend on the character of the problem. In other words the structure of the functional interpolation function must satisfy compatibility and completeness requirements. More detail can be found in Heubner (1975). To satisfy these requirements the following linear interpolation functions are chosen:

$$\phi_i = \frac{\theta_j - \theta}{\theta_j - \theta_i}, \quad \phi_j = \frac{\theta - \theta_i}{\theta_j - \theta_i}. \tag{17}$$

To express external variable loads approximately in the element the following equations are used:

$$q = q_i\phi_i + q_j\phi_j, \quad m = m_i\phi_i + m_j\phi_j \tag{18}$$

where q and m are any components of the external load vectors **q** and **m**.

Also, to take into account variable cross-section, different rigidities defined in eqn (4) will be expressed in terms of interpolation functions as follows:

$$\frac{1}{EA} = A_i\phi_i + A_j\phi_j, \quad \frac{1}{GI_0} = I_i\phi_i + I_j\phi_j, \quad \frac{1}{EI_n} = X_i\phi_i + X_j\phi_j$$
$$\frac{1}{EI_b} = Y_i\phi_i + Y_j\phi_j, \quad \frac{1}{GA'} = K_i\phi_i + K_j\phi_j. \tag{19}$$

All expressions of unknown and known quantities in terms of interpolation functions are put into eqn (7) and after extremization of this functional with respect to 24 nodal variables, 24 element equations have been obtained (Table 1). During the mathematical manipulations the following properties of interpolation functions are used:

$$\int \phi_i \phi_i \, d\theta = \tfrac{1}{3}\Delta\theta, \quad \int \phi_j \phi_j \, d\theta = \tfrac{1}{3}\Delta\theta,$$

$$\int \phi_i \phi_j \, d\theta = \tfrac{1}{6}\Delta\theta, \quad \int \phi_i \phi_i' \, d\theta = -\tfrac{1}{2},$$

$$\int \phi_j \phi_j' \, d\theta = \tfrac{1}{2}, \quad \int \phi_i \phi_j' \, d\theta = -\tfrac{1}{2}. \tag{20}$$

The element equations are valid for three-dimensional bars with variable cross-sections. Their properties are:

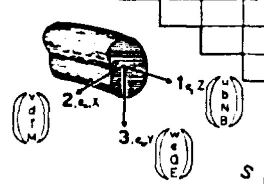
—the coefficient element matrix is symmetric;
—they reduce to plane equations of bars for $\tau = 0$;
—they give the straight bar equations for $\tau = 0$, $\kappa = 0$;
—for the beam with constant cross-section we have $A_i = A_j$, $I_i = I_j$, $X_i = X_j$, $Y_i = Y_j$, $K_i = K_j$;
—for special cases the size of matrix reduces. For plane problems the order of the matrix is 8×8;
—the numbers with hats are valid only when a dynamic condition is given for the corresponding node. Otherwise these numbers must be ignored.

The load vector has 24 elements, which are defined as follows:

Table 1. Element matrix

	u1	u2	v1	v2	w1	w2	b1	b2	d1	d2	e1	e2	N1	N2	T1	T2	Q1	Q2	B1	B2	M1	M2	E1	E2
u1													05·î	-05	tL/3	tL/6								
u2													05	-05·î	tL/6	tL/3								
v1						0							-tL/3	-tL/6	05·î	-05	rL/3	rL/6						
v2													-tL/6	-tL/3	05	-05·î	rL/6	rL/3						
w1															-rL/3	-rL/6	05·î	-05						
w2															-rL/6	-rL/3	05	-05·î						
b1																			05·î	-05	xL/3	xL/6		
b2																			05	-05·î	xL/6	xL/3		
d1																	L/3	L/6	-tL/3	-xL/6	05·î	-05	rL/3	rL/6
d2																	L/6	L/3	-tL/6	-xL/3	05	-05·î	rL/6	rL/3
e1															-L/3	-L/6					-rL/3	-rL/6	05·î	-05
e2															-L/6	-L/3					-rL/6	-rL/3	05	-05·î
N1													a1	a2										
N2														a3										
T1															k1	k2								
T2																k3								
Q1																	k1	k2						
Q2																		k3						
B1																			i1	i2				
B2																				i3				
M1																					x1	x2		
M2																						x3		
E1																							X	y2
E2																								y3

SYMMETRICAL



$A_i = 1/EA$

$I_i = 1/GI$. $Y_i = 1/EI_z$

$X_i = 1/EI_y$. $K_i = 1/GA'$

$a_1 = -L(A_i/4 + A_j/12)$. $a_2 = -L(A_i + A_j)/12$

$a_3 = -L(A_i/12 + A_j/4)$. $k_1 = -L(K_i/4 + K_j/12)$. $k_2 = -L(K_i + K_j)/12$

$k_3 = -L(K_i/12 + K_j/4)$. $i_1 = -L(I_i/4 + I_j/12)$. $i_2 = -L(I_i + I_j)/12$

$i_3 = -L(I_i/12 + I_j/4)$. $x_1 = -L(X_i/4 + X_j/12)$. $x_2 = -L(X_i + X_j)/12$

$x_3 = -L(X_i/12 + X_j/4)$. $y_1 = -L(Y_i/4 + Y_j/12)$. $y_2 = -L(Y_i + Y_j)/12$. $y_3 = -L(Y_i/12 + Y_j/4)$

$$
\begin{aligned}
&\frac{\partial I}{\partial u_i} = \ldots \mathrm{c}\left(\tfrac{1}{3}q_t^i + \tfrac{1}{6}q_t^j\right)\cdot\Delta\theta - \hat{N}_i, \quad \frac{\partial I}{\partial u_j} = \ldots \mathrm{c}\left(\tfrac{1}{6}q_t^i + \tfrac{1}{3}q_t^j\right)\cdot\Delta\theta - \hat{N}_j \\
&\frac{\partial I}{\partial v_i} = \ldots \mathrm{c}\left(\tfrac{1}{3}q_n^i + \tfrac{1}{6}q_n^j\right)\cdot\Delta\theta - \hat{T}_i, \quad \frac{\partial I}{\partial v_j} = \ldots \mathrm{c}\left(\tfrac{1}{6}q_n^i + \tfrac{1}{3}q_n^j\right)\cdot\Delta\theta - \hat{T}_j \\
&\frac{\partial I}{\partial w_i} = \ldots \mathrm{c}\left(\tfrac{1}{3}q_b^i + \tfrac{1}{6}q_b^j\right)\cdot\Delta\theta - \hat{Q}_i, \quad \frac{\partial I}{\partial w_j} = \ldots \mathrm{c}\left(\tfrac{1}{6}q_b^i + \tfrac{1}{3}q_b^j\right)\cdot\Delta\theta - \hat{Q}_j \\
&\frac{\partial I}{\partial b_i} = \ldots \mathrm{c}\left(\tfrac{1}{3}m_t^i + \tfrac{1}{6}m_t^j\right)\cdot\Delta\theta - \hat{B}_i, \quad \frac{\partial I}{\partial b_j} = \ldots \mathrm{c}\left(\tfrac{1}{6}m_t^i + \tfrac{1}{3}m_t^j\right)\cdot\Delta\theta - \hat{B}_j \\
&\frac{\partial I}{\partial d_i} = \ldots \mathrm{c}\left(\tfrac{1}{3}m_n^i + \tfrac{1}{6}m_n^j\right)\cdot\Delta\theta - \hat{M}_i, \quad \frac{\partial I}{\partial d_j} = \ldots \mathrm{c}\left(\tfrac{1}{6}m_n^i + \tfrac{1}{3}m_n^j\right)\cdot\Delta\theta - \hat{M}_j \\
&\frac{\partial I}{\partial e_i} = \ldots \mathrm{c}\left(\tfrac{1}{3}m_b^i + \tfrac{1}{6}m_b^j\right)\cdot\Delta\theta - \hat{E}_i, \quad \frac{\partial I}{\partial e_j} = \ldots \mathrm{c}\left(\tfrac{1}{6}m_b^i + \tfrac{1}{3}m_b^j\right)\cdot\Delta\theta - \hat{E}_j \\
&\frac{\partial I}{\partial N_i} = \ldots \hat{u}_i, \quad \frac{\partial I}{\partial N_j} = \ldots \hat{u}_j, \quad \frac{\partial I}{\partial T_i} = \ldots \hat{v}_i \\
&\frac{\partial I}{\partial T_j} = \ldots \hat{v}_j, \quad \frac{\partial I}{\partial Q_i} = \ldots \hat{w}_i, \quad \frac{\partial I}{\partial Q_j} = \ldots \hat{w}_j \\
&\frac{\partial I}{\partial B_i} = \ldots \hat{b}_i, \quad \frac{\partial I}{\partial B_j} = \ldots \hat{b}_j, \quad \frac{\partial I}{\partial M_i} = \ldots \hat{d}_i \\
&\frac{\partial I}{\partial M_j} = \ldots \hat{d}_j, \quad \frac{\partial I}{\partial E_i} = \ldots \hat{e}_i, \quad \frac{\partial I}{\partial E_j} = \ldots \hat{e}_j.
\end{aligned}
\tag{21}
$$

Quantities with a hat are singular known nodal values. When this is not given, the corresponding value in the coefficient matrix must be ignored, as stated above. The other quantities are nodal values of external continuous loads. The concentrated external loads are taken into account by the Lagrange multiplier method. The inclusion of boundary condition terms in the functional (7) plays a very important role in the application. In classical finite element applications, the element equation corresponding to the known nodal value is excluded by the coding system. In this study it is proposed that all equations must be included. In this case the known values of nodal variables are imposed by Lagrange multipliers. The inclusion of Lagrange multipliers in the solution process provides the equality of the number of equations and the number of unknowns. The physical interpretation of Lagrange multipliers may be obtained by the boundary condition terms of the functionals (7). For example, let us assume the deflection of any nodal point is known to be zero. To impose this value, the deflection is multiplied by λ and

$$\lambda v = 0.$$

This is written as an expression included in the functional (7). The physical meaning of λ corresponds to the shear force at this node.

4. APPLICATIONS

Test case 1: the cantilever beam

To illustrate the theory and to show the accuracy of the method a simple cantilever beam with a concentrated load will be solved first. Only a single element will be used. The nodal variables are shown in Fig. 2. The boundary conditions are

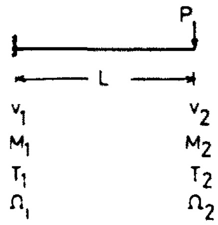


Fig. 2. Nodal unknowns of cantilever beam.

$$v_1 = 0, \quad \Omega_1 = 0, \quad (T_2 - P) = 0, \quad M_2 = 0.$$

We can add these boundary conditions to the functional (7) by Lagrange multipliers. Using the element equation we obtain

$$\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2 - \lambda_1 = 0$$

$$\frac{L}{3}T_1 + \frac{L}{6}T_2 + \tfrac{1}{2}M_1 - \tfrac{1}{2}M_2 - \lambda_2 = 0$$

$$\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2 + \frac{L}{3}\Omega_1 + \frac{L}{6}\Omega_2 - \frac{L}{3GA'}T_1 - \frac{L}{6GA'}T_2 = 0$$

$$\tfrac{1}{2}\Omega_1 + \tfrac{1}{2}\Omega_2 - \frac{L}{3EI}M_1 - \frac{L}{6EI}M_2 = 0$$

$$\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2 + T_2 - P = 0$$

$$\frac{L}{6}T_1 + \frac{L}{3}T_2 + \tfrac{1}{2}M_1 - \tfrac{1}{2}M_2 - M_2 = 0$$

$$-\tfrac{1}{2}v_1 - \tfrac{1}{2}v_2 + \frac{L}{6}\Omega_1 + \frac{L}{3}\Omega_2 - \frac{L}{6GA'}T_1 - \frac{L}{3GA'}T_2 + v_2 + \lambda_3 = 0$$

$$-\tfrac{1}{2}\Omega_1 - \tfrac{1}{2}\Omega_2 - \frac{L}{6EI}M_1 - \frac{L}{3EI}M_2 + \Omega_2 + \lambda_4 = 0.$$

v_1, Ω_1, T_2 and M_2 are already known as boundary conditions. The solutions of these equations are:

$$\lambda_1 = \lambda_2 = 0, \quad T_1 = P, \quad M_1 = -PL, \quad v_2 + \lambda_3 = \frac{PL^3}{3EI} + \frac{PL}{GA'}, \quad \Omega_2 + \lambda_4 = -\frac{PL^2}{2EI}.$$

The interesting result here is that all values of variables are exact. For the same problem by Kirchhoff analysis 75% accuracy has been obtained by Bathe (1982). Prathap *et al.* (1986), with their approach using four elements, have reached exact results for cantilever beam's tip displacement and tip rotation. They did not give any results for internal forces. As is mentioned above, using the new method with one element, complete results, displacement and internal forces have been obtained exactly for a cantilever beam. These comparisons show the power of the method.

Test case 2: pinched ring

An exact solution for the radial deflection under load P and for the bending moment, shear force and axial force at any station ϕ from the vertical can be easily derived from

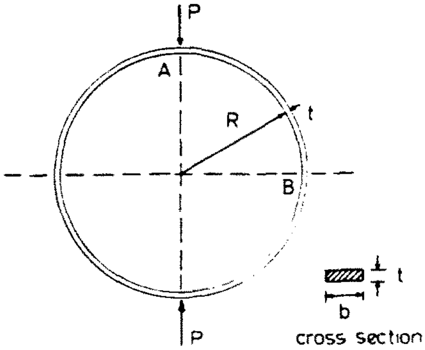


Fig. 3. Beam with fixed ends and external load P.

Table 2. Results of test case 2 by FFM

Number of elements	2		3		Exact	
Points	A	B	A	B	A	B
w	−3.40	−2.40	−3.18	3.18	−3.16	2.90
M	1789	−1017	1778	−1015	1782	−1017
N	0	−222	0	−222	0	−223
T	−222	0	−226	0	−223	0

Table 3. Results of test case 2 by conventional coding

Number of elements	3		4		6	
Points	A	B	A	B	A	B
w	−3.16	3.02	−3.16	2.87	−3.15	2.88
M	1668	−938	1717	−1131	1751	−1065
N	4	−222	2.6	−222	1.12	−222
T	−224	0	−226	0	−223	0

elementary energy principles (Fig. 3). For a ring with $R = 12.58$ cm, $t = 0.24$ cm, $b = 2.54$ cm, $E = 7.24 \times 10^6$ N cm^{-2}, $\nu = 0.3125$ and $P = 444.98$ N. These values are taken from Bathe (1982). As is seen from Table 2, the internal forces are almost exact. The same problem can be solved by a conventional coding system using the same functional but without Lagrange multipliers; the results are given in Table 3.

Comparisons show that boundary condition terms play a very important role in the results. Even if the same functional is used in both methods, the convergence of the solution using FFM is much faster than conventional finite element approaches.

Babu and Prathap (1986) did not give numerical results but instead illustrated their findings diagrammatically. They reported that CMCS elements give satisfactory results. There is not enough information, such as the number of unknowns and numerical results, to compare the two approaches. Using this approach, three-dimensional problems, such as a helix, can be solved easily.

5. CONCLUSIONS

In order to develop a mixed finite element model which can be applied to most general bar structures, it is first necessary to establish a new variational functional $I[y]$. Boundary conditions are included in this functional, which can be used for both the new FFM and conventional finite element formulations. These boundary condition terms are very effective in numerical solutions, especially in redundant problems. The proposed method appears to be a more efficient formulation than conventional finite element formulations, and can be easily applied to different types of structures from straight bars to three-dimensional structures.

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APPENDIX

To obtain functionals for field equations (1)–(4), the functional analysis method has been invoked. The matrix form of the field equation is :

$$
\left[\begin{array}{cccccc|cccc}
0 & 0 & 0 & -\frac{d}{ds} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{d}{ds} & -\mathbf{t}\times & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{d}{ds} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\frac{d}{ds} & \mathbf{t}\times & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \mathbf{D} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & \mathbf{C} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{array}\right]
\left\{\begin{array}{c}
\mathbf{u} \\ \boldsymbol{\Omega} \\ \mathbf{M} \\ \mathbf{T} \\ \boldsymbol{\omega} \\ \boldsymbol{\gamma} \\ \hline \mathbf{u}_0 \\ \boldsymbol{\Omega}_0 \\ \mathbf{M}_0 \\ \mathbf{T}_0
\end{array}\right\}
=
\left\{\begin{array}{c}
\mathbf{q} \\ \mathbf{m} \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline \hat{\mathbf{T}}_0 \\ \hat{\mathbf{M}}_0 \\ \hat{\boldsymbol{\Omega}}_0 \\ \hat{\mathbf{u}}_0
\end{array}\right\}. \tag{A1}
$$

This matrix can be represented by the following operator form,

$$
\begin{aligned}
\mathscr{L}\mathbf{y} &= \mathbf{f} \\
\mathbf{Q} &= \mathscr{L}\mathbf{y} - \mathbf{f}
\end{aligned} \tag{A2}
$$

where

$$
\mathbf{f}:\begin{cases}\text{The load vector has two kinds of elements:}\\ \text{external loads and known values of the boundary variables}\end{cases}
$$

$$
\mathbf{y}:\begin{cases}\text{The unknown vector has two kinds of variables:}\\ \text{domain variables and boundary variables}\end{cases}
$$

$\mathscr{L}$: The coefficient matrix.

If $\mathbf{Q}$ is a potential operator, the equality

$$
\langle d\mathbf{Q}(\mathbf{y}, \bar{\mathbf{y}}), \mathbf{y}^* \rangle = \langle d\mathbf{Q}(\mathbf{y}, \mathbf{y}^*), \bar{\mathbf{y}} \rangle \tag{A3}
$$

must be satisfied (Oden *et al.*, 1976). $d\mathbf{Q}(\mathbf{y}, \bar{\mathbf{y}})$ and $d\mathbf{Q}(\mathbf{y}, \mathbf{y}^*)$ are the Gâteaux derivatives of the operator in the $\bar{\mathbf{y}}$ and $\mathbf{y}^*$ directions, which are constant elements in the domain. Gâteaux derivatives of the operator are defined as

$$\mathrm{d}\mathbf{Q}(\mathbf{y},\bar{\mathbf{y}}) = \left.\frac{\partial \mathbf{Q}(\mathbf{y}+\tau\bar{\mathbf{y}})}{\partial \tau}\right|_{\tau=0} \tag{A4}$$

where τ is a scaler. Using these definitions the open form of the inner product is

$$\begin{aligned}[\mathrm{d}\mathbf{Q}(\mathbf{y},\bar{\mathbf{y}}),\mathbf{y}^*] = {} & \left[\frac{\mathrm{d}\bar{\mathbf{T}}}{\mathrm{d}s},\mathbf{u}^*\right] - \left[\left(\frac{\mathrm{d}\bar{\mathbf{M}}}{\mathrm{d}s}+\mathbf{t}\times\bar{\mathbf{T}}\right),\mathbf{\Omega}^*\right] + \left[\left(\frac{\mathrm{d}\bar{\mathbf{\Omega}}}{\mathrm{d}s}-\bar{\omega}\right),\mathbf{M}^*\right] + \left[\left(\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}s}+\mathbf{t}\times\bar{\mathbf{\Omega}}-\bar{\gamma}\right),\mathbf{T}^*\right] \\ & +[(-\bar{\mathbf{M}}+\mathbf{D}\bar{\omega}),\omega^*]+[(-\bar{\mathbf{T}}+\mathbf{C}\bar{\gamma}),\gamma^*]+[\bar{\mathbf{T}}_0,\mathbf{u}_0^*]_\varepsilon+[\bar{\mathbf{M}}_0,\mathbf{\Omega}_0^*]_\sigma-[\bar{\mathbf{\Omega}}_0,\mathbf{M}_0^*]_\varepsilon-[\bar{\mathbf{u}}_0,\mathbf{T}_0^*]_\varepsilon .\end{aligned} \tag{A5}$$

After some simple manipulations it can be shown that the equality (A3) holds and the operator $\mathbf{Q}$ is a potential operator. Since the operator is potential then the functional corresponding to the field equations is obtained as (Oden and Reddy, 1976)

$$I(\mathbf{y}) = \int_0^1 [\mathbf{Q}(s\mathbf{y},\mathbf{y}),\mathbf{y}]\,\mathrm{d}s \tag{A6}$$

where s is a scaler quantity. Two types of functional $I(\mathbf{y})$ can be obtained after some maniupulations as

$$\begin{aligned}I_1(\mathbf{y}) = {} & -\left[\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s}+\mathbf{t}\times\mathbf{\Omega}\right),\mathbf{T}\right]-[\mathbf{q},\mathbf{u}]+\tfrac{1}{2}\left[\mathbf{M},\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}s}\right]-[\mathbf{m},\mathbf{\Omega}]+\left[\left(\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}s}-\omega\right),\mathbf{M}\right]-\tfrac{1}{2}[\gamma,\mathbf{T}]-\tfrac{1}{2}[(\mathbf{M}-\mathbf{D}\omega),\omega] \\ & -\tfrac{1}{2}[(\mathbf{T}-\mathbf{C}\gamma),\gamma]-[\hat{\mathbf{T}},\mathbf{u}]_\sigma-[\hat{\mathbf{M}},\mathbf{\Omega}]_\sigma-\tfrac{1}{2}[(\mathbf{u}-\hat{\mathbf{u}}),\mathbf{T}]_\varepsilon-\tfrac{1}{2}[(\mathbf{\Omega}-\hat{\mathbf{\Omega}}),\mathbf{M}]_\varepsilon\end{aligned} \tag{A7}$$

and

$$\begin{aligned}I_2(\mathbf{y}) = {} & -\left[\mathbf{u},\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right]+[(\mathbf{t}\times\mathbf{\Omega}),\mathbf{T}]-[\mathbf{q},\mathbf{u}]+\tfrac{1}{2}\left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s},\mathbf{\Omega}\right]-[\mathbf{m},\mathbf{\Omega}]-\tfrac{1}{2}[\omega,\mathbf{M}]-\tfrac{1}{2}[\gamma,\mathbf{T}]-\tfrac{1}{2}[(\mathbf{M}-\mathbf{D}\omega),\omega] \\ & -\tfrac{1}{2}[(\mathbf{T}-\mathbf{C}\gamma),\gamma]+[(\mathbf{T}-\hat{\mathbf{T}}),\mathbf{u}]_\sigma+[(\mathbf{M}-\hat{\mathbf{M}}),\mathbf{\Omega}]_\sigma+\tfrac{1}{2}[\hat{\mathbf{u}},\mathbf{T}]_\varepsilon+\tfrac{1}{2}[\hat{\mathbf{\Omega}},\mathbf{M}]_\varepsilon .\end{aligned} \tag{A8}$$